Corner Ranking, Realizable Vectors, and Extremal Cop-Win Graphs

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Abstract

We investigate certain kinds of extremal graphs related to the game of cops and robber. We focus on cop-win graphs, that is, graphs on which a single cop can catch the robber. The capture time of a cop-win graph is the minimum number of moves the cop needs to win. We consider graphs that are extremal with respect to capture time, i.e. their capture time is as large as possible given their order. We give a new characterization of the set of extremal graphs. For our alternative approach we assign a rank to each vertex of a graph, and then study which configurations of ranks are possible. We partially determine which configurations are possible, enough to prove some further extremal results. A full classification of such configurations is motivated by the insight such a classification would give into how the capture time of a graph relates to its structure; we leave that project as an open question.

1 Introduction

The game of cops and robber is a perfect-information two-player pursuit-evasion game played on a graph. To begin the game, the cop and robber each choose a vertex to occupy, with the cop choosing first. Play then alternates between the cop and the robber, with the cop moving first. On a turn a player may move to an adjacent vertex or stay still. If the cop and robber ever occupy the same vertex, the robber is caught and the cop wins. If the cop can force a win on a graph, we say the graph is **cop-win**. The game was introduced by Nowakowski and Winkler [5], and Quilliot [9]. A nice introduction to the game and its many variants is found in the book by Bonato and Nowakowski [1].

One of the fundamental results about the game is a characterization of the cop-win graphs as those graphs which have a **cop-win ordering** [5], [9]. Independently, Clarke, Finbow, and MacGillivray [3] and the authors of this paper [6] developed an alternative characterization that we call *corner ranking*. As with cop-win orderings, corner ranking characterizes which graphs are cop-win. Corner ranking can also be used to determine the capture time of a cop-win graph, as well as describe optimal strategies (in terms of capture time) for the cop and robber, where the **capture time** of a cop-win graph is the fewest number of moves the cop needs to guarantee a win, not counting her initial placement (for example, on the path with 5 vertices, the capture time is 2). In Section 2, we describe the corner ranking procedure and some useful properties of it that were proved in [6].

Bonato et. al. [2] make the following interesting definition.

Definition 1.1. *Suppose $n > 0$ is a natural number. Let $\mathsf{capt}(n)$ = the capture time of a* cop-win *graph on n vertices with maximum capture time.*

For example, $\mathsf{capt}(4) = 2$ since a path on four vertices has capture time 2, and no graph with 4 vertices has a capture time greater than 2. Define a cop-win graph with n vertices to be **CT-maximal** if no other cop-win graph on n vertices has a larger capture time. Building on [2], Gavenciak [4] proved the that for $n \geq 7$, $\mathsf{capt}(n) = n - 4$, and gave a characterization of the CT-maximal graphs. The proof relies on detailed analysis of the conceivable cop and robber strategies, and uses a computer search at one step. In Theorem 4.3, we use corner rank to give a different proof, one which avoids a computer search.

Our basic approach to the proofs is to associate cop-win graphs with vectors, where by a **vector** we simply mean a finite list of integers. The corner ranking procedure assigns each vertex in a cop-win graph an integer, so in Section 3 we define the *rank cardinality vector* of a cop-win graph as the vector whose i^{th} entry is the number of vertices of corner rank i. Since the length of the vector is the corner rank of the graph, which determines capture time, we can characterize the CT-maximal graphs by determining which vectors are *realizable*, i.e. which vectors are the rank cardinality vector for some cop-win graph. Thus the fundamental issue in our paper becomes determining which vectors are realizable and which are not.

In Section 3 we determine enough about the realizability of vectors to prove Theorem 4.3. In Section 5 we turn to the general question of realizability; we motivate this question by showing how understanding realizability helps us understand the structure of the following interesting class of graphs.

Definition 1.2. *Let $\mathcal{G}_n^t$ be the set of cop-win graphs with n vertices and capture time t.*

In Section 5.1 we prove more about realizability, enough to allow us to characterize $\mathcal{G}_n^{n-5}$ in Section 5.2. Our two main theorems are Theorem 4.3 and Theorem 5.11, characterizing $\mathcal{G}_n^{n-4}$ and $\mathcal{G}_n^{n-5}$, respectively. In Section 5.3 we investigate more complicated realizable vectors with the intention of providing some direction on the general question of classifying all the realizable vectors.

2 Corner Ranking

In this section we state the definitions and theorems about corner rank that are necessary for this paper. For a full development including proofs and examples, see [6]. In this paper all graphs are finite and non-empty, i.e. they have at least one vertex; all numbers are integers. We follow a typical Cops and Robber convention by assuming that all graphs are reflexive, that is *all graphs have a loop at every vertex so that a vertex is always adjacent to itself*; we will never draw or mention such edges. This assumption simplifies much of the following discussion (for example when we define homomorphism) while leaving the game play unchanged. For a graph $\mathbf{G}$, $V(\mathbf{G})$ refers to the vertices of $\mathbf{G}$ and $E(\mathbf{G})$ refers to the edges of $\mathbf{G}$. If $\mathbf{G}$ is a graph and X is a vertex or set of vertices in $\mathbf{G}$, then by $\mathbf{G} - X$ we mean the subgraph of $\mathbf{G}$ induced by $V(\mathbf{G}) \setminus X$. We say that a vertex u **dominates** a set of vertices X if u is adjacent to every vertex in X. Given a vertex v in a graph, by $\mathsf{N}[v]$, the **closed neighborhood of v**, we mean the set of vertices adjacent to v. For distinct vertices v and w, if $\mathsf{N}[v] \subseteq \mathsf{N}[w]$ then we say that v is a **corner** and that w **corners** v; if $\mathsf{N}[v] \subsetneq \mathsf{N}[w]$, we say that v is a **strict corner** and that w **strictly corners** v; if $\mathsf{N}[v] = \mathsf{N}[w]$, we call v and w **twins**.

A **cop-win ordering** of a graph (also called a **dismantling ordering**) [5, 9] is produced by removing one corner at a time, until all the vertices have been removed or there is no corner to remove. As a small but significant modification of the cop-win ordering, rather than removing one corner at a time, we remove all the current strict corners simultaneously, assigning them a number we call the *corner rank*.

Definition 2.1 (Corner Ranking Procedure). *For any graph $\mathbf{G}$, we define a corresponding **corner rank** function, $\mathbf{cr}$, which maps each vertex of $\mathbf{G}$ to a positive integer or ∞. We also define a sequence of associated graphs $\mathbf{G}^{[1]}, \ldots, \mathbf{G}^{[\alpha]}$.*

0. Initialize $\mathbf{G}^{[1]} = \mathbf{G}$, and $k = 1$.

1. If $\mathbf{G}^{[k]}$ is a clique, then:

- *Let $\mathbf{cr}(x) = k$ for all $x \in \mathbf{G}^{[k]}$.*
- *Then* **stop**.

2. Else if $\mathbf{G}^{[k]}$ is not *a clique and has* no *strict corners, then:*

- *Let $\mathbf{cr}(x) = \infty$ for all $x \in \mathbf{G}^{[k]}$.*
- *Then* **stop**.

3. Else:

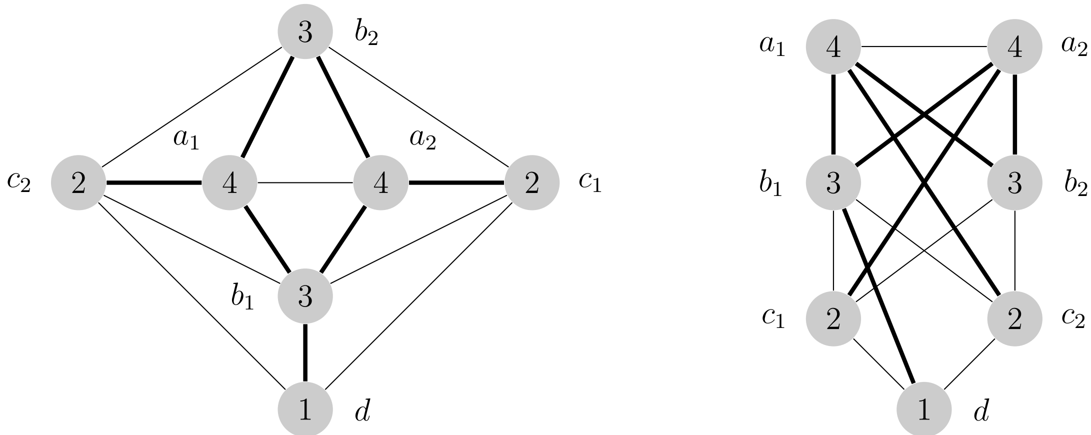

Figure 1: Two representations of the graph $\mathbf{H}_7$

- *Let V be the set of strict corners in $\mathbf{G}^{[k]}$.*
- *For all $x \in V$, let $\mathbf{cr}(x) = k$.*
- *Let $\mathbf{G}^{[k+1]} = \mathbf{G}^{[k]} - V$.*
- *Increment k by 1 and return to Step 1.*

*Define the **corner rank** of $\mathbf{G}$, denoted $\mathbf{cr}(\mathbf{G})$, to be the same as the vertex of $\mathbf{G}$ with largest corner rank; we understand ∞ to be larger than all integers.*

In [6], we show that the corner ranking procedure is well-defined. As an example, we apply the corner ranking procedure to the graph $\mathbf{H}_7$ in Figure 1; this graph was introduced in [2], more typically drawn like the graph on the left. The corner ranking procedure begins by assigning the strict corner d rank 1. After d is removed, c_1 and c_2 are strict corners, and are thus assigned corner rank 2. Likewise, b_1 and b_2 are assigned corner rank 3. After b_1 and b_2 are removed, the remaining vertices, a_1 and a_2, form a clique and so are assigned corner rank 4; thus the corner rank of the graph is 4.

Convention. *In all the figures, when a vertex w has rank k and is strictly cornered in $\mathbf{G}^{[k]}$ by a vertex v of higher rank, we draw the edge vw with a thick line. Also, the number drawn inside a vertex indicates its corner rank.*

As another example, consider Figure 2. While $\mathbf{cr}(x) = 1$ and $\mathbf{cr}(y) = 2$, once x and y have been removed there are no strict corners, and what remains is not a clique, so the other 5 vertices have corner rank ∞; thus the graph has corner rank ∞.

Convention. *Since graphs with corner rank 1 are a cliques, and thus a trivial case, we will assume all graphs have corner rank at least 2*

For cop-win graphs, the capture time depends on a structural property of the highest ranked vertices.

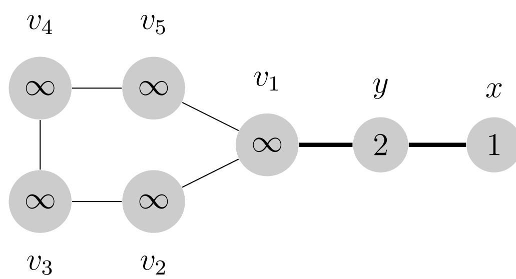

Figure 2: The corner ranking of a non cop-win graph.

Definition 2.2. *Suppose* $\mathbf{G}$ *is a graph with a* finite *corner rank* α. *We say that* $\mathbf{G}$ *is a* 1***-cop-win graph*** *if one of the two equivalent conditions holds:*

- *Some vertex of corner rank* α *dominates* $V(\mathbf{G}^{[\alpha-1]})$.
- *Every vertex of corner rank* α *dominates* $V(\mathbf{G}^{[\alpha-1]})$.

Otherwise we say $\mathbf{G}$ *is a* 0***-cop-win graph***.

We now state the main result (Theorem 6.1) of [6], which relates the corner rank of a graph to its capture time; for a graph $\mathbf{G}$, we let $\mathsf{capt}(\mathbf{G})$ = the capture time of $\mathbf{G}$ (note that the capt function is overloaded so that it makes sense for a graph or an integer as input; see Definition 1.1).

Theorem 2.3 ([6], Theorem 6.1)**.** *A graph is cop-win if and only if it has finite corner rank. Furthermore, for an* r*-cop-win-graph* $\mathbf{G}$, $\mathsf{capt}(\mathbf{G}) = \mathbf{cr}(\mathbf{G}) - r$.

For example, the graph $\mathbf{H}_7$ in Figure 1 is 1-cop-win with corner rank 4, so it is cop-win with capture time $4 - 1 = 3$. In Figure 2, the graph is not cop-win and has corner rank ∞.

In [6], Lemma 2.3, we prove the following useful technical property; we will use this property so often that we will refer to it by a name: *Upward Cornering.*

Lemma 2.4 (*Upward Cornering*)**.** *If a vertex* v *has corner rank* k *in a graph* $\mathbf{G}$ *of rank larger than* k*, then* v *is strictly cornered in* $\mathbf{G}^{[k]}$ *by a vertex of higher rank.*

3 Rank Cardinality Vectors and Realizability

Convention. *For the remainder of the paper, we assume* $\mathbf{G}$ *is a cop-win graph with finite corner rank* $\alpha \geq 2$.

For $1 \leq k \leq \alpha$, let $\mathsf{V}_k^{(\mathbf{G})}$ denote the set of vertices of rank k in $\mathbf{G}$; we just write V_k if the graph is apparent from context. By the term **vector**, we mean a finite list of positive integers.

A vector $\overline{\mathbf{x}} = (x_\alpha, x_{\alpha-1}, \ldots, x_1)$ has **length** α and **sum** $(x_\alpha + \cdots + x_1)$. We use typical conventions for representing vectors, writing, for example, $2, \ldots, 2$ to mean a list of some number of 2's (*at least one*). In this section, we introduce the **rank cardinality vector** of a cop-win graph, which is a vector whose entries correspond to the number of vertices of each rank.

Definition 3.1. *The **rank cardinality vector** of a graph* $\mathbf{G}$ *is the vector* $(x_\alpha, x_{\alpha-1}, \ldots, x_1)$*, where* $x_k = |V_k|$.

Definition 3.2. *A vector* $\overline{\mathbf{x}} = (x_\alpha, x_{\alpha-1}, \ldots, x_1)$ *is **realizable** if it is the rank cardinality vector of some cop-win graph* $\mathbf{G}$*. We say that* $\mathbf{G}$ ***realizes*** $\overline{\mathbf{x}}$*, or that* $\overline{\mathbf{x}}$ *is **realized by*** $\mathbf{G}$*. For* $r \in \{0, 1\}$*,* $\overline{\mathbf{x}}$ *is* r***-realizable** if there is an* r*-cop-win graph* $\mathbf{H}$ *that realizes it. We say that* $\mathbf{H}$ ***r-realizes*** $\overline{\mathbf{x}}$*, or that* $\overline{\mathbf{x}}$ *is **r-realized by*** $\mathbf{H}$.

For example, the graph $\mathbf{H}_7$ in Figure 1 realizes $(2, 2, 2, 1)$, so since $\mathbf{H}_7$ is a 1-cop-win graph, (2,2,2,1) is 1-realizable. We will see that some vectors are *not* realizable. Since an r-realizable vector with sum n and length α corresponds to an r-cop-win graph on n vertices with capture time $\alpha - r$, to understand $\mathcal{G}_n^{\alpha-r}$, to determine $\mathsf{capt}(n)$, and to answer related questions, the following question is of fundamental interest.

Question 3.3. *For* $r \in \{0, 1\}$*, which vectors are* r*-realizable?*

In this section, we answer this question to the extent necessary to give a proof of Theorem 4.3. In Section 5, we develop this question further and explore the general issue of realizability.

3.1 Augmentations, Initial Segments, and Extensions

We introduce three ways to alter a realizable vector to obtain another realizable vector: taking an augmentation, initial segment, or standard extension.

Definition 3.4. *Consider a vector* $(x_\alpha, \ldots, x_1)$.

- *If the vector* $(y_\alpha, \ldots, y_1)$ *has the property that* $x_i \leq y_i$ *for all* $1 \leq i \leq \alpha$*, we say that* $(y_\alpha, \ldots, y_1)$ *is an **augmentation** of* $(x_\alpha, \ldots, x_1)$.
- *For* $k \geq 1$*, any vector of the form* $(x_\alpha, \ldots, x_k)$ *is called an **initial segment** of* $(x_\alpha, \ldots, x_1)$.
- *Any vector of the form* $(x_\alpha, \ldots, x_1, z_1, z_2, \ldots, z_l)$ *is called an **extension** of* $(x_\alpha, \ldots, x_1)$*. If* $z_i = x_1$ *for all* $1 \leq i \leq l$*, it is called a **standard extension**.*
- *For all the notions (augmentation, initial segment, extension, and standard extension), we include the trivial case in which the vector is unchanged.*
- *We say that* $\overline{\mathbf{x}} \leq \overline{\mathbf{y}}$ *if* $\overline{\mathbf{y}}$ *is an augmentation (possibly trivial) of a standard extension (possibly trivial) of* $\overline{\mathbf{x}}$.

For example, a standard extension of $(3, 2, 2)$ is $(3, 2, 2, 2, 2)$ and an augmentation of $(3, 2, 2, 2, 2)$ is $(5, 2, 6, 2, 3)$, so $(3, 2, 2) \leq (5, 2, 6, 2, 3)$.

Lemma 3.5. *If a vector is r-realizable, then so is any augmentation of it.*

Proof. It suffices to show that if $\overline{\mathbf{x}} = (x_\alpha, \ldots, x_1)$ is r-realizable, then so is $\overline{\mathbf{y}} = (y_\alpha, \ldots, y_1)$, where for some k, $y_k = x_k + 1$, and for $j \neq k$, $y_j = x_j$. Consider a graph $\mathbf{G}$ which r-realizes $\overline{\mathbf{x}}$. Choose a vertex $v \in \mathsf{V}_k$, and let $\mathbf{G}'$ be the graph obtained by adding a twin of v to $\mathbf{G}$. Then $\mathbf{G}'$ r-realizes the vector $\overline{\mathbf{y}}$. □

Lemma 3.6. *If a vector is r-realizable, then so is any initial segment.*

Proof. If $\mathbf{G}$ r-realizes the vector $(x_\alpha, \ldots, x_1)$, then the initial segment $(x_\alpha, \ldots, x_k)$ is realized by $\mathbf{G}^{[k]}$. □

Lemma 3.7. *Suppose $\overline{\mathbf{x}} = (x_\alpha, \ldots, x_1)$ and $\overline{\mathbf{y}} = (x_\alpha, \ldots, x_1, y_k, \ldots, y_1)$ is a standard extension. If $\overline{\mathbf{x}}$ is r-realizable then so is $\overline{\mathbf{y}}$. Moreover, if $\mathbf{H}$ realizes $\overline{\mathbf{x}}$, then there is a graph $\mathbf{G}$ realizing $\overline{\mathbf{y}}$ such that $\mathbf{G}^{[k+1]} = \mathbf{H}$.*

Proof. It suffices to show that if $\overline{\mathbf{x}} = (x_\alpha, \ldots, x_1)$ is r-realized by $\mathbf{H}$, then $(x_\alpha, \ldots, x_1, x_1)$ is r-realized by some $\mathbf{G}$ where $\mathbf{G}^{[2]} = \mathbf{H}$. Suppose $\mathbf{H}$ r-realizes $\overline{\mathbf{x}}$ with rank 1 vertices $v_1, \ldots, v_{x_1}$. Let $\mathbf{G}$ be the graph obtained by adding the following to $\mathbf{H}$: vertices $w_1, \ldots, w_{x_1}$ and edges $v_1w_1, \ldots, v_{x_1}w_{x_1}$. Then the vertices $w_1, \ldots, w_{x_1}$ are the only strict corners in $\mathbf{G}$, the rank cardinality vector of $\mathbf{G}$ is $(x_\alpha, \ldots, x_1, x_1)$, and $\mathbf{G}^{[2]} = \mathbf{H}$. □

From Lemmas 3.5 and 3.7, we conclude the following.

Corollary 3.8. *For two vectors $\overline{\mathbf{x}}$ and $\overline{\mathbf{y}}$ where $\overline{\mathbf{x}} \leq \overline{\mathbf{y}}$, if $\overline{\mathbf{x}}$ is r-realizable, then $\overline{\mathbf{y}}$ is r-realizable.*

As a special case, note that if $\overline{\mathbf{x}} = (x_\alpha, \ldots, x_1)$ is r-realizable and $x_1 = 1$, then any extension of $\overline{\mathbf{x}}$ is r-realizable. We will often use the contrapositive form of Corollary 3.8: If $\overline{\mathbf{x}} \leq \overline{\mathbf{y}}$, and $\overline{\mathbf{y}}$ is not r-realizable, then $\overline{\mathbf{x}}$ is not r-realizable. For example, in Corollary 3.24 we show that for any k, the vector $(1, 3, k, 1)$ is not realizable, which also implies that any vector of the form $(1, 2, k, 1)$ is not realizable.

3.2 Projections and Path Contraction

Lemma 2.4 allows us to define what we call projection functions. Again, we quote the relevant content from [6], though here we assume the graphs have finite corner rank, thus simplifying the definitions. We write $f : \mathbf{H} \longrightarrow \mathbf{G}$ to mean that f is a function whose domain is the **non-empty subsets** of $V(\mathbf{H})$ and whose codomain is the **non-empty subsets** of $V(\mathbf{G})$.

Definition 3.9. *Suppose $\mathbf{G}$ is a graph with finite corner rank α. We define the functions $f_1, \ldots, f_{\alpha-1}$ and $F_1, \ldots, F_{\alpha-1}, F_\alpha$, where $f_k : \mathbf{G}^{[k]} \longrightarrow \mathbf{G}^{[k+1]}$ and $F_k : \mathbf{G} \longrightarrow \mathbf{G}^{[k]}$.*

- *For a single vertex $u \in V(\mathbf{G}^{[k]})$, define:*

$$f_k(\{u\}) = \begin{cases} \{u\} & \text{if } \mathbf{cr}(u) > k \\ \text{the set of vertices in } \mathbf{G}^{[k+1]} \text{ that strictly corner } u \text{ in } \mathbf{G}^{[k]} & \textbf{otherwise} \end{cases}$$

- $f_k(\{u_1, \ldots, u_t\}) = \bigcup_{1 \le i \le t} f_k(\{u_i\})$

- *Let $F_1 : \mathbf{G} \longrightarrow \mathbf{G}$ be the identity function*

- *For $1 < k \le \alpha$, let $F_k = f_{k-1} \circ \cdots \circ f_1$*

For a function h whose domain is sets of vertices, we adopt the usual convention that $h(u) = h(\{u\})$ for a single vertex u. We say v is a k**-projection** (or simply a **projection**) of w if $v \in F_k(w)$. The key property of the projection functions, proved in [6], is that they are homomorphisms.

Definition 3.10. *Given two graphs $\mathbf{H}$ and $\mathbf{G}$, and a function $h : V(\mathbf{H}) \longrightarrow V(\mathbf{G})$, we say that h is a **homomorphism** if for vertices $u, v \in V(\mathbf{H})$ and vertices $u^* \in h(u), v^* \in h(v)$:*

$$u \text{ is adjacent to } v \text{ implies } u^* \text{ is adjacent to } v^*.$$

Lemma 3.11. *[6] Given a graph $\mathbf{G}$, its associated functions f_k and F_k are homomorphisms.*

Definition 3.12. *Suppose $\mathbf{G}$ is a graph, and $\mathbf{H}$ is a subgraph of $\mathbf{G}$, where the vertices of $\mathbf{H}$ are $\{v_1, \ldots, v_k\}$. A k**-projection of** $\mathbf{H}$ is a graph induced by a set of vertices $\{v'_1, \ldots, v'_k\}$, where v'_i is a k-projection of v_i.*

We will often refer to the k-projections of paths. We denote a path with m vertices by $\mathbf{P}_m$, say it has length $m-1$, and represent it by the vertices $(v_1, \ldots, v_m)$ where for $1 \le i < m$, v_i is adjacent to v_{i+1}. Lemma 3.13 follows directly from Lemma 3.11.

Lemma 3.13. *Let $\mathbf{G}$ be a graph with rank α and $\mathbf{P} = (v_1, v_2, \ldots, v_m)$ be a path in $\mathbf{G}$. For all $1 \le k \le \alpha$, every k-projection of $\mathbf{P}$ contains a path (possibly a single vertex) from $v'_1 \in F_k(v_1)$ to $v'_m \in F_k(v_m)$ whose length is at most $m-1$.*

The following corollary will be used so often, we will refer to applications of it by the name *Path Contraction.*

Corollary 3.14 (*Path Contraction*)**.** *If v and w are vertices in $\mathbf{G}$ of rank k where the shortest path from v to w in $\mathbf{G}^{[k]}$ has length m, then there is* no *path from v to w in $\mathbf{G}$ of length less than m.*

Corollary 3.14 will be used as a tool to show many configurations are impossible. For example, if v and w are nonadjacent vertices of rank k without a common neighbor of rank k or higher, they cannot have a common neighbor at all.

3.3 Vectors: Realizable and Not Realizable

We now prove a number of results about realizability: some showing that a particular kind of vector is realizable, some showing that a particular kind of vector is not realizable, and some placing restrictions on the structure of graphs realizing particular kinds of vectors.

Lemma 3.15. *Suppose v is a vertex of rank $k > 1$. Then for every vertex w that strictly corners v in $\mathbf{G}^{[k]}$, v must have a neighbor of rank $k-1$ that is* not *adjacent to w.*

Proof. If not, then there is a vertex w that strictly corners v in $\mathbf{G}^{[k-1]}$, contradicting the assumption that v has rank k. □

Corollary 3.16. *In a graph with rank α, every vertex of rank $k > 1$ has at least one neighbor of rank $k-1$. In particular, if there is exactly one vertex v of rank k, for some $k < \alpha$, then v is adjacent to all the vertices of rank $k+1$.*

Lemma 3.17. *In a graph with rank α, no vertex of rank $\alpha - 1$ can dominate $\mathsf{V}_{\alpha-1}$.*

Proof. Suppose some vertex b of rank $\alpha - 1$ dominates $\mathsf{V}_{\alpha-1}$. By *Upward Cornering*, let a be a vertex of rank α that strictly corners b in $\mathbf{G}^{[\alpha-1]}$. Then a must also dominate $\mathsf{V}_{\alpha-1}$, making $\mathbf{G}$ 1-cop-win. In a 1-cop-win graph, every vertex of rank α dominates $\mathsf{V}_{\alpha-1}$, and so b is adjacent to every vertex of rank α. Thus b is adjacent to every vertex in $\mathbf{G}^{[\alpha-1]}$, contradicting the assumption that a strictly corners b in $\mathbf{G}^{[\alpha-1]}$. □

Corollary 3.18. *No vector $(x_\alpha, \dots, x_1)$ with $x_{\alpha-1} = 1$ is realizable.*

Lemma 3.19. *No vector $(x_\alpha, \dots, x_1)$ with $x_{\alpha-2} = 1$ is realizable.*

Proof. Suppose $\mathbf{G}$ is a graph realizing $(x_\alpha, x_{\alpha-1}, \dots, x_1)$, where $x_{\alpha-2} = 1$, and c is the unique vertex of rank $\alpha - 2$. By Corollary 3.16, $\mathsf{V}_{\alpha-1} \subseteq \mathsf{N}[c]$. By *Upward Cornering*, some vertex x of rank at least $\alpha - 1$ strictly corners c in $\mathbf{G}^{[\alpha-2]}$. If $x \in \mathsf{V}_{\alpha-1}$, then x dominates $\mathsf{V}_{\alpha-1}$, which contradicts Lemma 3.17. If $x \in \mathsf{V}_\alpha$, then x is adjacent to every vertex in $\mathbf{G}^{[\alpha-2]}$. Thus $\mathbf{G}^{[\alpha-2]}$ has rank at most 2, which contradicts the assumption that $\mathbf{G}^{[\alpha-2]}$ has rank 3. □

While the set of realizable vectors includes vectors that are not 0-realizable, the set of realizable vectors is in fact the same as the set of 1-realizable vectors.

Lemma 3.20. *Every realizable vector is 1-realizable.*

Proof. Suppose $\overline{\mathbf{x}} = (x_\alpha, \dots, x_1)$ is a realizable vector, realized by $\mathbf{G}$. If $x_\alpha = 1$, then $\mathbf{G}$ must be 1-cop-win so $\overline{\mathbf{x}}$ is 1-realizable (though not 0-realizable). Suppose $x_\alpha > 1$. By Corollary 3.18 and Lemma 3.19: $x_{\alpha-1}, x_{\alpha-2} > 1$. By Lemma 3.5, it suffices to show that we can 1-realize (2), $(2,2)$, $(2,2,2)$, and any vector of the form $(2,2,2,1,\dots,1)$. Since all of these vectors are initial segments or standard extensions of $(2,2,2,1)$, which is realized by the 1-cop-win graph $\mathbf{H}_7$ (see Figure 1), they are all 1-realizable. □

We remark that in the proof of the next theorem, as well as in the proofs of Theorems 4.3 and 5.10, we claim that certain small vectors are uniquely realized. For example, in proving Part (i) of the next theorem, we show that $(1,2,2)$ is uniquely realized by $\mathbf{P}_5$. However, in Part (iii) we claim *without proof* that $(2,2,2)$ is uniquely realized by $\mathbf{P}_6$, and in Theorems 4.3 and 5.10 we make similar claims without proof. We omit these proofs because they are short technical proofs. However, we invite the interested reader to check the claims, which invariably use some combination of the following: *Upward Cornering*, *Path Contraction*, Lemma 3.15, Corollary 3.16, and Lemma 3.17. To see drawings of graphs that realize various small vectors see section 5 of [7].

Theorem 3.21.

(i) The vector $(1,2,\ldots,2)$ *of length* α *is uniquely realized by* $\mathbf{P}_{2\alpha-1}$*.*

(ii) The vector $(1,2,\ldots,2,1)$ *is* not *realizable.*

(iii) The vector $(2,\ldots,2)$ *of length* α *is uniquely 0-realized by* $\mathbf{P}_{2\alpha}$*.*

(iv) The vector $(2,\ldots,2,1)$ *is* not *0-realizable.*

Proof.

Proof of (i): The statement is true by inspection for $\alpha = 1, 2$. It is clear that $\mathbf{P}_{2\alpha-1}$ realizes $(1,2,\ldots,2)$; we proceed by induction, with base case $\alpha = 3$, to show the uniqueness.

Base case ($\alpha = 3$): Consider any graph $\mathbf{G}$ realizing $(1,2,2)$; suppose $\mathsf{V}_3 = \{a\}$, $\mathsf{V}_2 = \{b_1, b_2\}$, and $\mathsf{V}_1 = \{c_1, c_2\}$. The vector $(1,2)$ is uniquely realized by $\mathbf{P}_3$, so b_1 and b_2 are not adjacent. If they are both adjacent to c_1, then by *Upward Cornering* a must strictly corner c_1. In order for b_1 and b_2 to not be strictly cornered by a in $\mathbf{G}$, they must each be adjacent to c_2 and a must not. But then no vertex of rank 2 or 3 strictly corners c_2, contradicting *Upward Cornering*. Thus each vertex of rank 2 has a unique neighbor of rank 1, so we assume that $b_1c_1, b_2c_2 \in E(\mathbf{G})$, while $b_1c_2, b_2c_1 \notin E(\mathbf{G})$. By Lemma 3.15, a cannot be adjacent to either c_1 or c_2, and thus for $i = 1, 2$, by Lemma 2.4, c_i must be strictly cornered by b_i. Thus $c_1c_2 \notin E(\mathbf{G})$, and $\mathbf{G} = \mathbf{P}_5$.

Inductive step: Now consider a graph $\mathbf{G}$ with rank $\alpha \geq 4$ realizing the vector $(1,2,\ldots,2)$. By the inductive hypothesis, $\mathbf{G}^{[2]} = \mathbf{P}_{2\alpha-3} = (v_1, v_2, \ldots, v_{2\alpha-3})$. Since $\alpha \geq 4$, the shortest path in $\mathbf{G}^{[2]}$ between v_1 and $v_{2\alpha-3}$ (which are the two rank 2 vertices in $\mathbf{G}$) has length at least four. Let y and z be the two rank 1 vertices in $\mathbf{G}$ (see Figure 3). By Lemma 3.15, v_1 and $v_{2\alpha-3}$ must each be adjacent to some rank 1 vertex. However, by *Path Contraction*, v_1 and $v_{2\alpha-3}$ cannot both be adjacent to the same rank 1 vertex in $\mathbf{G}$, and furthermore, y and z cannot be adjacent, or else there is a path of length 2 or 3 between v_1 and $v_{2\alpha-3}$ in $\mathbf{G}$. Thus

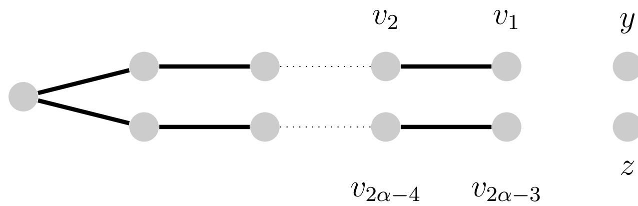

Figure 3: The unique graph realizing $(1, 2, \ldots, 2)$ is $\mathbf{P}_{2\alpha-1}$.

without loss of generality, assume $yv_1, zv_{2\alpha-3} \in E(\mathbf{G})$ and $zv_1, yv_{2\alpha-3} \notin E(\mathbf{G})$. To show that $\mathbf{G} = \mathbf{P}_{2\alpha-1}$ we just need to rule out edges of the form yv_i, where v_i has rank at least 3 (an analogous discussion holds of z). Suppose there is an edge $yv_i \in E(\mathbf{G})$ where v_i has rank at least 3. Then the vertex that strictly corners y in $\mathbf{G}$ is not v_1, but must be adjacent to v_1, and so must be v_2. But in this case v_2 strictly corners v_1 in $\mathbf{G}$, contradicting the assumption that v_1 has rank 2. So no edges from higher rank vertices to y or z are possible, and $\mathbf{G} = \mathbf{P}_{2\alpha-1}$.

Proof of (ii): Corollary 3.18 and Lemma 3.19 imply that $(1, 1)$ and $(1, 2, 1)$ are not realizable. For $\alpha \geq 4$, if $\mathbf{G}$ is a graph realizing $(x_\alpha, \ldots, x_1)$ with $x_\alpha = x_1 = 1$ and $x_k = 2$ for $2 \leq k < \alpha$, then by (i), $\mathbf{G}^{[2]} = \mathbf{P}_{2\alpha-3}$ and the two rank 2 vertices u and v in $\mathbf{G}$ have distance $2\alpha - 4 \geq 4$ in $\mathbf{G}^{[2]}$. If there were one vertex of rank 1, then by Corollary 3.16 the rank 1 vertex is adjacent to both u and v, yielding a length 2 path from u to v, contradicting *Path Contraction*.

Proof of (iii): This proof is almost the same as the proof of (i), but now with a base case stating that $(2, 2, 2)$ is uniquely 0-realized by $\mathbf{P}_6$; the proof of the base case is a similar technical proof to that of the base case for $(1, 2, 2)$.

Proof of (iv): This proof is the same as the proof of (ii), using (iii) instead of (i).

$\square$

We now turn our attention to graphs with rank 4, starting with a simple technical lemma.

Lemma 3.22. *If a graph realizes $(a, b, c, 1)$ then there is a vertex of rank 3 or 4 that dominates the rank 2 vertices.*

Proof. Let $\mathbf{G}$ be the graph and let d be the lone vertex in V_1. By Corollary 3.16, $\mathsf{V}_2 \subseteq \mathsf{N}[d]$. By *Upward Cornering*, some vertex x of rank greater than 1 must strictly corner d, so $\mathsf{V}_2 \subseteq \mathsf{N}[x]$. If $x \in \mathsf{V}_2$, then by *Upward Cornering* let y be a vertex of rank at least 3 that strictly corners x in $\mathbf{G}^{[2]}$, otherwise let $y = x$. In either case, we have a vertex y in either V_3 or V_4 such that $\mathsf{V}_2 \subseteq \mathsf{N}[y]$. $\square$

Theorem 3.23. *Suppose a graph realizes* $(1, m, k, 1)$*. Then the subgraph induced by the rank* 3 *vertices is connected.*

Proof. Let $\mathbf{G}$ be the graph and let $\mathbf{H}$ be the subgraph induced by the rank 3 vertices. Assume for the sake of contradiction that the claim is false. Suppose a is the rank 4 vertex, two components of $\mathbf{H}$ have vertex sets B_1 and B_2, and for $i = 1, 2$, $b_i \in B_i$. By Lemma 3.15, there must be a rank 2 vertex c_1 adjacent to b_1 but not to a. Since b_1 is only adjacent to rank 3 vertices in B_1, by *Upward Cornering*, c_1 must be strictly cornered in $\mathbf{G}^{[2]}$ by a vertex in B_1 and thus c_1 is only adjacent to rank 3 vertices in B_1. Similarly, there is a rank 2 vertex c_2 that is adjacent to b_2, but not to a; likewise, c_2 is only adjacent to rank 3 vertices in B_2. If c_1 and c_2 are adjacent or have a common neighbor c of rank 2 then the vertex of higher rank (which we have by *Upward Cornering*) that strictly corners c (or c_1 if c_1 and c_2 are adjacent) in $\mathbf{G}^{[2]}$ would have to be adjacent to both c_1 and c_2. However, no such higher rank vertex exists since it would have to be in both B_1 and B_2, but these sets are disjoint. Thus c_1 and c_2 are at distance at least three in $\mathbf{G}^{[2]}$, and by *Path Contraction*, they cannot both be adjacent to the single rank 1 vertex, contradicting Corollary 3.16. $\square$

Since the graph induced by the rank 1 vertices of any graph realizing $(1, 3)$ is not connected, Lemma 3.23 implies the following corollary.

Corollary 3.24. *For all* $k \geq 1$*, the vector* $(1, 3, k, 1)$ *is* not *realizable.*

Theorem 3.25.

(i) *For* $k \geq 1$*,* $(2, 4, k, 1)$ *is* not 0*-realizable.*

(ii) $(2, 5, 2, 1)$ *is* not 0*-realizable.*

Proof. The proofs of (i) and (ii) are similar, with only some differences at the end. Consider for the sake of contradiction a graph $\mathbf{G}$ that 0-realizes $(2, m, k, 1)$ or $(2, 5, 2, 1)$. Since $\mathbf{G}$ is a 0-cop-win graph and $\mathsf{V}_4 = \{a_1, a_2\}$ has only two vertices, there are rank 3 vertices b_1 and b_2 such that $a_1b_1, a_2b_2 \in E(\mathbf{G})$ and $a_1b_2, a_2b_1 \notin E(\mathbf{G})$. *For* $i = 1, 2$*,* a_i *must strictly corner* b_i *and every rank 3 neighbor of* b_i *in* $\mathbf{G}^{[3]}$*; we will use this point throughout the proof.* Since no rank 4 vertex is adjacent to both b_1 and b_2, they can share no common neighbors in $\mathbf{G}^{[3]}$ (since no rank 4 vertex could corner such a vertex in $\mathbf{G}^{[3]}$), and by *Path Contraction*, b_1 and b_2 must be at distance at least 3 in $\mathbf{G}$. For $i = 1, 2$, let c_i be a rank 2 vertex adjacent to b_i but not a_i, which must exist by Lemma 3.15. Since the distance between b_1 and b_2 is at least 3, c_1 and c_2 must be distinct vertices, and $b_1c_2, b_2c_1 \notin E(\mathbf{G})$.

Since no vertex of rank 4 dominates V_2, by Lemma 3.22, there is a vertex b_3 of rank 3 that dominates V_2, and b_3 is not b_1 or b_2. Without loss of generality suppose a_2 corners b_3 in $\mathbf{G}^{[3]}$. Now consider what corners c_1 in $\mathbf{G}^{[2]}$: neither a_i, not b_2 because it is not adjacent to c_1, and neither b_1 nor b_3 since that would force b_1 and b_3 to be neighbors and would imply a_2 is adjacent to b_1, a contradiction. So a fourth distinct rank 3 vertex b_4 must corner c_1 in $\mathbf{G}^{[2]}$, and thus b_4 must be adjacent to both b_1 and b_3.

To finish the proof for (i): Now consider what vertex of rank at least 3 strictly corners c_2 in $\mathbf{G}^{[2]}$. Since the distance from b_1 to b_2 is at least 3, neither of b_1 or b_4 can be adjacent to b_2 and thus neither of these vertices can corner c_2. Neither vertex of rank 4 works since a_1 is not adjacent to b_2 and a_2 is not adjacent to c_2. So b_2 or b_3 strictly corners c_2 in $\mathbf{G}^{[2]}$, and are thus adjacent to each other. But now b_2 is strictly cornered by b_3 in $\mathbf{G}^{[2]}$, since they have the same neighbors in $\mathbf{G}^{[2]}$, except that b_3 is adjacent to c_1 and b_4, while b_2 is not.

To finish the proof for (ii): Since a_2 is not adjacent to b_1, a_1 must corner b_4 in $\mathbf{G}^{[3]}$, so in particular a_1 and b_4 are adjacent. Since b_1 is not strictly cornered by b_4 in $\mathbf{G}^{[2]}$, it must be adjacent to the fifth rank 3 vertex b_5, while b_4 and b_5 are not adjacent. Since b_4 corners c_1, b_5 is not adjacent to c_1, so by Lemma 3.15, b_5 must be adjacent to c_2. Since a_1 must strictly corner b_5 in $\mathbf{G}^{[3]}$, b_5 is not adjacent to b_2, and thus b_3 is the only vertex that can strictly corner c_2 in $\mathbf{G}^{[2]}$. But then b_3 is adjacent to the rank 3 vertices b_2, b_4, and b_5, and thus also a_1. Thus in $\mathbf{G}^{[3]}$, b_3 has at least the neighbors that a_2 has, contradicting the fact that a_2 strictly corners b_3 in $\mathbf{G}^{[3]}$. □

Theorem 3.26. *For any $m, k \geq 1$, $(m, 2, k, 1)$ is* not *0-realizable.*

Proof. For the sake of contradiction, suppose $\mathbf{G}$ 0-realizes $(m, 2, k, 1)$. Let $\mathsf{V}_3 = \{b_1, b_2\}$ and note that every rank 4 vertex is adjacent to exactly one of these two vertices. Thus $b_1b_2 \notin E(\mathbf{G})$, and these two vertices are at distance 3 in $\mathbf{G}^{[3]}$ and hence in $\mathbf{G}$. Thus by *Path Contraction* they share no rank two neighbors. By Lemma 3.22, there is a vertex x of rank 3 or 4 that dominates V_2. Since b_1 and b_2 must both have rank 2 neighbors but can't have any in common, neither of these vertices can be x. Thus x must be a rank 4 vertex. But if x is adjacent to b_i, then it strictly corners b_i in $\mathbf{G}^{[2]}$, contradicting the assumption that b_i has rank 3. □

Recall the graph $\mathbf{H}_7$ from Figure 1.

Lemma 3.27. *The vector (2,2,2,1) is uniquely realized by* $\mathbf{H}_7$.

Proof. Let $\mathbf{G}$ be a graph that realizes $(2, 2, 2, 1)$, with $\mathsf{V}_4 = \{a_1, a_2\}$, $\mathsf{V}_3 = \{b_1, b_2\}$, $\mathsf{V}_2 = \{c_1, c_2\}$, and $\mathsf{V}_1 = \{d\}$. Theorem 3.21 implies that $(2, 2, 2, 1)$ is not 0-realizable. Thus $\mathbf{G}^{[3]}$ must contain the edges a_1a_2, a_1b_1, a_2b_2, a_1b_2, a_2b_1, and since $\mathbf{G}^{[3]}$ is not a clique, there is not an edge b_1b_2. By Corollary 3.16, each of b_1 and b_2 must be adjacent to a vertex of rank 2, and Lemma 3.22 implies that some vertex x of rank 3 or 4 dominates V_2. If x were some a_i, then x would strictly corner each rank 3 vertex in $\mathbf{G}^{[2]}$, a contradiction. Thus without loss of generality we may assume b_1 dominates V_2 and both b_1 and b_2 are adjacent to c_1. Then only a vertex from V_4 can strictly corner c_1 in $\mathbf{G}^{[2]}$; without loss of generality, suppose a_2 is this vertex, so in particular, a_2 is adjacent to c_1. Since a_2 is not a dominating vertex in $\mathbf{G}^{[2]}$, it cannot be adjacent to c_2 and thus c_1 and c_2 cannot be adjacent. For a_2 not to strictly corner b_2 or a_1 in $\mathbf{G}^{[2]}$, each of these vertices must be adjacent to c_2, and a_1 cannot be adjacent to c_1 or else it dominates $\mathbf{G}^{[2]}$. Thus $\mathbf{G}^{[2]} = (\mathbf{H}_7)^{[2]}$.

By Corollary 3.16, the rank 1 vertex d is adjacent to both c_1 and c_2, which means it can only be strictly cornered by some b_i, without loss of generality, b_1. Since the rank 3 vertices

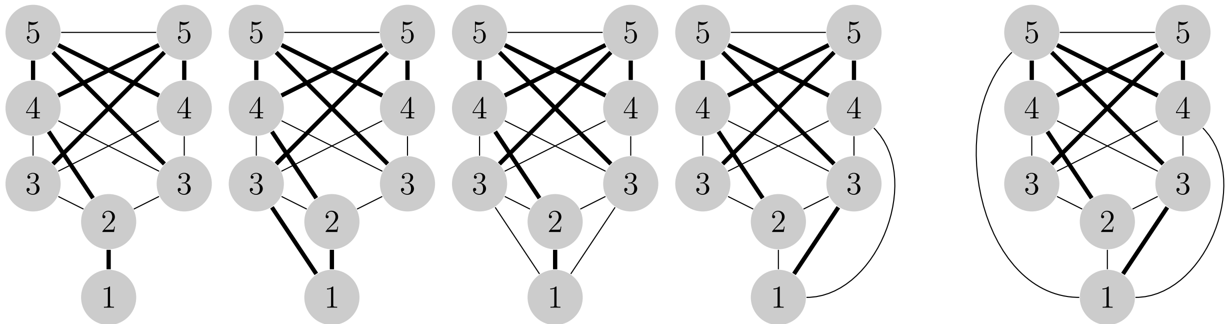

Figure 4: Some graphs in $\mathcal{H}_7^{+1}$

are not adjacent, d cannot be adjacent to b_2. Finally, by Lemma 3.15, d cannot be adjacent to any rank 4 vertex. Thus $\mathbf{G}$ is $\mathbf{H}_7$. □

4 A Characterization of the CT-Maximal Graphs

We can now characterize the rank cardinality vectors of all the CT-maximal graphs. The following definition will be used to classify the CT-maximal graphs having at least seven vertices.

Definition 4.1. *For $k \geq 0$, define $\mathcal{H}_7^{+k}$ to be a set of graphs that realize the length $4+k$ vector $(2,2,2,1,\ldots,1)$. Let $\mathcal{H}_7^{+}$ be $\bigcup_{k\geq 0}\mathcal{H}_7^{+k}$.*

For example, Lemma 3.27 implies that $\mathcal{H}_7^{+0} = \{\mathbf{H}_7\}$. Figure 4 displays some of the graphs in $\mathcal{H}_7^{+1}$. By Lemma 3.7, any standard extension of $(2,2,2,1)$ is realizable, so for each k, $\mathcal{H}_7^{+k}$ is non-empty. In [4], $\mathcal{M}$ is defined to be the set of CT-maximal graphs. We will see (in Theorem 4.3) that for $n \geq 9$, $\mathcal{H}_7^{+}$ is the same as $\mathcal{M}$. In Theorem 2 of [4] a nice, but somewhat involved characterization of $\mathcal{M}$ is given (stated to be true for $n \geq 8$, but actually true for $n \geq 9$). Our result gives a simpler characterization (for $n \geq 9$): A graph is in $\mathcal{M}$ exactly when it realizes $(2,2,2,1,\ldots,1)$. In the process of characterizing $\mathcal{M}$, Gavenciak [4] derives various properties of the graphs in $\mathcal{M}$; these properties follow almost immediately from our characterization of $\mathcal{M}$ by $\mathcal{H}_7^{+}$, summarized in the next theorem.

Theorem 4.2. *Suppose $\mathbf{G}$ is a graph on n vertices in $\mathcal{H}_7^{+}$. Then*

(i) $\mathbf{G}^{[\alpha-3]}$ *is* $\mathbf{H}_7$.

(ii) $\mathbf{G}$ *is 1-cop-win.*

(iii) $\mathsf{capt}(\mathbf{G}) = n-4$.

Proof. Property (i) follows from Lemma 3.27. Property (ii) follows from the fact that $\mathbf{H}_7$ is 1-cop-win. For Property (iii), note that $\mathbf{G}$ has rank $n-3$ and is 1-cop-win. Thus by Theorem 2.3, $\mathbf{G}$ has capture time $(n-3)-1 = n-4$. $\square$

The next theorem restates the main results of [4], with an alternative proof that does not use a computer search.

Theorem 4.3. *For $n \geq 7$, $\mathsf{capt}(n) = n-4$, and for graphs on at least 9 vertices, the CT-maximal graphs are exactly the graphs in $\mathcal{H}_7^+$. Furthermore, in Table 1, we describe $\mathsf{capt}(n)$ and the CT-maximal graphs for $n \leq 8$.*

n	$\mathsf{capt}(n)$	CT-Maximal Graphs with n vertices
1	0	$\mathbf{P}_1$
2	1	$\mathbf{P}_2$
3	1	$\mathbf{P}_3$, $\mathbf{K}_3$
4	2	$\mathbf{P}_4$
5	2	$\mathbf{P}_5$ and the 0-cop-win graphs realizing (2,3) and (3,2)
6	3	$\mathbf{P}_6$
7	3	$\mathbf{P}_7$, $\mathbf{H}_7$, and the 0-cop-win graphs realizing (2,2,3), (2,3,2), (3,2,2)
8	4	$\mathbf{P}_8$ and any graph in $\mathcal{H}_7^{+1}$

Table 1: CT-Maximal graphs with at most 8 vertices and their capture time

Proof. We begin with the case of $n \leq 8$, considering vectors realized by $\mathbf{P}_n$. By Theorem 3.21, when n is even, $\mathbf{P}_n$ is the unique 0-cop-win graph realizing the length $n/2$ vector $(2,\ldots,2)$, and when n is odd, $\mathbf{P}_n$ is the unique 1-cop-win graph realizing the length $\lceil n/2 \rceil$ vector $(1,2,\ldots,2)$. Thus when n is even, graphs whose rank cardinality vector has length less than $n/2$ cannot be CT-maximal, and when n is odd, graphs whose rank cardinality vector is less than $\lfloor n/2 \rfloor$ cannot be CT-maximal. Based on this observation, Table 2 lists all vectors with sum $n \leq 8$ that could possibly be the rank cardinality vector of some CT-maximal graph; by Corollary 3.18 and Lemma 3.19, we exclude the vectors whose second or third entry is 1. Note that the first vector (in **bold**) is the rank cardinality vector for the corresponding path $\mathbf{P}_n$.

To prove the theorem for $n \leq 8$, it suffices to show that each vector is either: 1) not realizable, 2) has capture time less than that of $\mathbf{P}_n$, or 3) is accounted for in Table 1. We proceed by cases on the values of $n \leq 8$, employing Theorem 2.3 and using the immediate fact that if the first entry is 1, then a graph that realizes the vector must be 1-cop-win. Also, recall the remarks in the paragraph directly before Theorem 3.21, where we discuss the issue of uniquely realizing small vectors and why we omit some proofs. At various points in this proof all we need to show is that some vector is realizable; in some of those cases, as an interesting tangent, we claim that the vector is uniquely realized, or we produce all the graphs realizing the vector.

n	Vectors
1	**(1)**
2	**(2)**
3	**(1,2)**, (3)
4	**(2,2)**, (1,3)
5	**(1,2,2)**, (1,4), (3,2), (2,3)
6	**(2,2,2)**, (1,3,2), (1,2,3), (1,2,2,1)
7	**(1,2,2,2)**, (2,2,2,1), (2,2,3), (2,3,2), (3,2,2), (1,2,2,1,1), (1,3,2,1), (1,2,3,1)
8	**(2,2,2,2)**, (2,2,2,1,1)

Table 2: Vectors with sum $n \leq 8$ and length at least $\lfloor n/2 \rfloor$

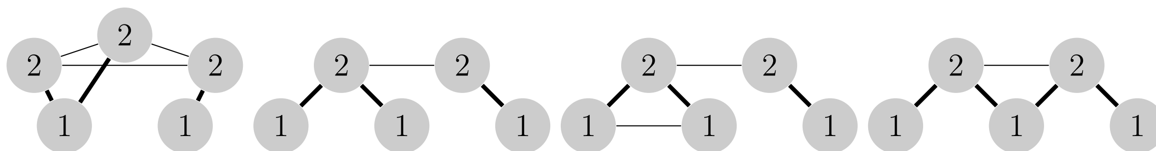

Figure 5: The unique graph 0-realizing $(3, 2)$ and the three graphs 0-realizing $(2, 3)$

- For $n = 1, 2, 3$ all the vectors listed in Table 2 have corresponding graphs listed in Table 1.
- For $n = 4$, a graph realizing $(1, 3)$ has capture time $1 < 2$, so it is not CT-maximal.
- For $n = 5$, besides $(1, 2, 2)$, the vectors listed in Table 2 have length less than 3, so they can only have capture time 2 if they are 0-cop-win, which means we also get as CT-maximal graphs the unique graph 0-realizing $(3, 2)$ and the three graphs 0-realizing $(2, 3)$. (See Figure 5.)
- For $n = 6$, the only vector, besides $(2, 2, 2)$, corresponding to a capture time of 3 or greater is $(1, 2, 2, 1)$, but that vector is not realizable, by Theorem 3.21.
- For $n = 7$, the vector $(2, 2, 2, 1)$ is uniquely realized by $\mathbf{H}_7$, using Lemma 3.27. To achieve the required capture time of 3, we can also take one of the three graphs 0-realizing $(2, 2, 3)$ or one of the unique graphs 0-realizing $(2, 3, 2)$, or $(3, 2, 2)$. (See Figure 6.) The rest of the vectors are not realizable: $(1, 2, 2, 1, 1)$ is not realizable by Theorem 3.21, and $(1, 3, 2, 1)$ and $(1, 2, 3, 1)$ are not realizable by Corollary 3.24.
- For $n = 8$, by definition, the vector $(2, 2, 2, 1, 1)$ is only realized by graphs from $\mathcal{H}_7^{+1}$.

Now we consider $n \geq 9$. We show that $\mathbf{H}_7^{+(n-7)}$ contains all the CT-maximal graphs. For $\mathbf{H}_7^{+(n-7)}$ not to contain all the CT-maximal graphs we would need a realizable vector $\overline{\mathbf{x}} = (x_\alpha, \ldots, x_1)$ besides $(2, 2, 2, 1, 1, \ldots, 1)$ with one of the following properties.

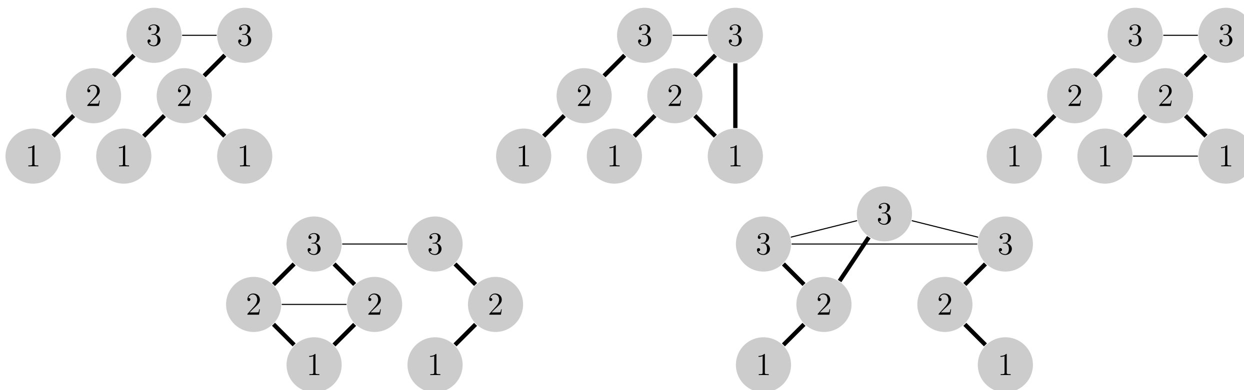

Figure 6: **Top**: The three graphs 0-realizing $(2, 2, 3)$. **Bottom**: The unique graphs 0-realizing $(2, 3, 2)$ and $(3, 2, 2)$

- Type 0: $\alpha \geq n - 4$ and $\overline{\mathbf{x}}$ is 0-realizable.
- Type 1: $\alpha \geq n - 3$ and $\overline{\mathbf{x}}$ is 1-realizable.

We show that no such vectors are realizable. Keep in mind that in both cases $x_{\alpha-1}$ and $x_{\alpha-2}$ must be at least 2 by Corollary 3.18 and Lemma 3.19.

We rule out the type 0 vectors. Let $\overline{\mathbf{y}} = (y_\alpha, \ldots, y_1)$ be the vector $(2, 2, 2, 1, \ldots, 1)$. Being 0-realizable, $x_\alpha \geq 2$. Since $\alpha \geq n - 4$, such an $\overline{\mathbf{x}}$ would be an augmentation of $\overline{\mathbf{y}}$ where all entries of $\overline{\mathbf{x}}$ are the same as the entries of $\overline{\mathbf{y}}$ with the possible exception of one entry of $\overline{\mathbf{y}}$, which is one larger than its corresponding entry in $\overline{\mathbf{x}}$. No matter where the 1 is added, or if nothing is added, one of the following vectors must be an initial segment of $\overline{\mathbf{x}}$: (3,2,2,1), (2,3,2,1), (2,2,3,1), (2,2,2,1) or (2,2,2,2,1). The first and third vectors are not 0-realizable by Theorem 3.26, and the second is not 0-realizable by Theorem 3.25; the last two vectors are not 0-realizable by Theorem 3.21.

Now we rule out the type 1 vectors. Let $\overline{\mathbf{y}} = (y_\alpha, \ldots, y_1)$ be the vector $(1, 2, 2, 1, \ldots, 1)$. Since $\alpha \geq n-3$, such an $\overline{\mathbf{x}}$ would be an augmentation of $\overline{\mathbf{y}}$ where all entries of $\overline{\mathbf{x}}$ are the same as the entries of $\overline{\mathbf{y}}$ with the possible exception of one entry of $\overline{\mathbf{y}}$, which is one larger than its corresponding entry in $\overline{\mathbf{x}}$. The value 1 cannot be added to y_α since that would mean $\mathbf{G}$ is in $\mathcal{H}_7^+$. No matter where else 1 is added, or if nothing is added, one of the following vectors must be an initial segment of $\overline{\mathbf{x}}$: (1,2,2,1), (1,2,2,2,1), (1,3,2,1), (1,2,3,1). By Theorem 3.21 the first two vectors are not realizable, and by Corollary 3.24, the last two vectors are not realizable. $\square$

5 General Results on Realizability

The goal of this section is to begin an investigation into answering Question 3.3: Which vectors are realizable? Before partially answering this question, we motivate the question by discussing how it relates to understanding the set of graphs $\mathcal{G}_n^t$ from Definition 1.2. Let $\mathcal{V}_n^\alpha$ be the realizable vectors of length α with sum n. One way to specify which graphs are in $\mathcal{G}_n^t$ is simply as follows: All the 0-realizable graphs with a rank cardinality vector in $\mathcal{V}_n^t$, together with all the 1-realizable graphs with a rank cardinality vector in $\mathcal{V}_n^{t+1}$. We now reduce Question 3.3 to a more specific question (recall the ordering from Definition 3.4).

Definition 5.1. *A vector $\overline{\mathbf{x}}$, of length at least 2, is r-**minimal** if the only r-realizable vector $\leq \overline{\mathbf{x}}$, of length at least 2, is $\overline{\mathbf{x}}$ itself. A vector is **minimal** if it is either 0-minimal or 1-minimal.*

For example, it follows from Theorem 4.3 that $(2,2,2,1)$ is 1-minimal, and thus, for example, $(2,7,2,1)$ and $(2,2,2,1,1,1)$ are *not* 1-minimal.

Definition 5.2. *Suppose V is a set of vectors and $\overline{\mathbf{x}}$ is a vector. We say that $\overline{\mathbf{x}} \leq V$ (respectively $V \leq \overline{\mathbf{x}}$) if there is a vector $\overline{\mathbf{y}} \in V$ such that $\overline{\mathbf{x}} \leq \overline{\mathbf{y}}$ (respectively $\overline{\mathbf{y}} \leq \overline{\mathbf{x}}$).*

Definition 5.3. *A pair of sets $(\mathcal{B}_0, \mathcal{B}_1)$ is a **base** for $\mathcal{G}_n^t$ if $\mathcal{B}_r$ consists of all the r-minimal vectors $\leq \mathcal{V}_n^{t+r}$.*

For example, by Theorem 4.3, for $n \geq 9$, $(\emptyset, \{(2,2,2,1)\})$ is a base for $\mathcal{G}_n^{n-4}$, and in Theorem 5.11, we will prove that for $n \geq 11$, $(\{(3,3,2,1)\}, \{(2,2,2,1),(1,4,2,1)\})$ is a base for $\mathcal{G}_n^{n-5}$.

Lemma 5.4. *Suppose $\mathcal{G}_n^t$ has base $(\mathcal{B}_0, \mathcal{B}_1)$. Then $\mathcal{G}_n^t$ consists of the graphs that*

- 0-*realize a vector* $\overline{\mathbf{y}} \in \mathcal{V}_n^t$ *such that* $\mathcal{B}_0 \leq \overline{\mathbf{y}}$, or
- 1-*realize a vector* $\overline{\mathbf{y}} \in \mathcal{V}_n^{t+1}$ *such that* $\mathcal{B}_1 \leq \overline{\mathbf{y}}$.

For example, by Theorem 4.3, for $n \geq 9$, $\mathcal{G}_n^{n-4}$ is exactly the set of graphs that 1-realize a vector $\overline{\mathbf{y}}$ such that $(2,2,2,1) \leq \overline{\mathbf{y}}$, and $\overline{\mathbf{y}}$ has sum n and length $n-3$. The following lemma follows from the fact that no two minimal vectors are comparable.

Lemma 5.5. *For any integers n and t such that $1 \leq t \leq n-4$, $\mathcal{G}_n^t$ has a unique base.*

Determining which vectors are minimal is sufficient to determine the base for any $\mathcal{G}_n^t$, giving us a characterization of these graphs in the manner indicated by Lemma 5.4. Thus we can make Question 3.3 more specific:

Question 5.6. *For any $\mathcal{G}_n^t$, determine its base. To do this reduces to the question: Which vectors are minimal?*

The base is unique, but furthermore, for a typical case we consider, it is finite, independent of n, and in fact stabilizes once n is large enough.

Lemma 5.7. *For any positive integer c, if $m, n \geq 2c+1$, then the base of $\mathcal{G}_n^{n-c}$ is the same as the base of $\mathcal{G}_m^{m-c}$, and the base contains a finite number of vectors. Moreover, any vector in the base has length at most $c+1$.*

Proof. Suppose $\overline{\mathbf{x}}$ is in the base of $\mathcal{G}_n^{n-c}$, so $\overline{\mathbf{x}}$ is r-minimal, for some r, and there is some vector $\overline{\mathbf{y}} \in \mathcal{V}_n^{n-c+r}$ such that $\overline{\mathbf{x}} \leq \overline{\mathbf{y}}$. Note that the sum of $\overline{\mathbf{y}}$ is exactly $c-r$ more than its length, so it can have at most $c-r$ entries that are larger than 1, a property that thus must also be true of $\overline{\mathbf{x}}$. Thus since $n \geq 2c+1$, if $r=0$ it cannot be the case that $(2,2) \leq \overline{\mathbf{x}}$, and if $r=1$ it cannot be the case that $(1,2) \leq \overline{\mathbf{x}}$. Thus $\overline{\mathbf{x}}$ must be a vector of the form

$$(x_1, x_2, \ldots, x_k, 1), \text{ where } x_i \geq 2 \text{ for } i \geq 2.$$

Furthermore, since $\overline{\mathbf{x}}$ can have at most $c-r$ entries that are larger than 1, we have that $k \leq c$, and since $\overline{\mathbf{x}} \leq \overline{\mathbf{y}}$, the sum of $\overline{\mathbf{x}}$ is at most $2c+1$. There are only a finite number of vectors with length at most $c+1$ and sum at most $2c+1$, and thus the base is finite. Further, since the last entry of $\overline{\mathbf{x}}$ is 1, every extension of $\overline{\mathbf{x}}$ is realizable, and thus if $\overline{\mathbf{x}} \leq \mathcal{V}_n^{n-c+r}$ for some n, then for all $m \geq n$, $\overline{\mathbf{x}} \leq \mathcal{V}_m^{m-c+r}$, so the base of $\mathcal{G}_n^{n-c}$ is contained in the base of $\mathcal{G}_m^{m-c}$. Conversely if $\overline{\mathbf{x}} = (x_1, x_2, \ldots, x_k, 1)$, is a base vector for $\mathcal{G}_m^{m-c}$, so as argued above, $k \leq c$ and the sum of $\overline{\mathbf{x}}$ is at most $2c+1$, then it can be checked that, as long as $n \geq 2c+1$, we have $\overline{\mathbf{x}} \leq \mathcal{V}_n^{n-c+r}$, so $\overline{\mathbf{x}}$ is a base vector for $\mathcal{G}_n^{n-c}$. □

Now we know that any $\mathcal{G}_n^{n-c}$ has a unique fixed finite base for sufficiently large n. With some work, we actually found such bases for $\mathcal{G}_n^{n-4}$ and $\mathcal{G}_n^{n-5}$. We are left with the apparently difficult question of finding such a base for other values of c.

Thus we have motivated the program of determining which vectors are minimal as a way to understand which graphs on a given number of vertices have a given capture time. In Subsection 5.1 we list some minimal vectors. In Subsection 5.2 we use some of the minimal vectors from Subsection 5.1 to characterize the graphs on n vertices with capture time $n-5$, i.e. the graphs with a capture time one less than the maximum possible. Then in Subsection 5.3, we begin a more general investigation of finding the minimal vectors.

5.1 Minimal Vectors

We begin the program of finding minimal vectors.

Theorem 5.8. *The following vectors are* 1*-minimal.*

1. $(1,2)$
2. $(1,4,2,1)$
3. $(2,2,2,1)$

Proof.

1. The path $\mathbf{P}_3$ 1-realizes $(1,2)$. Corollary 3.18 implies $(1,1)$ is not realizable, so $(1,2)$ is minimal.
2. See Figure 7 for a graph that 1-realizes $(1,4,2,1)$. Corollary 3.24 and Lemma 3.19 imply $(1,3,2,1)$ and $(1,4,1)$ are not realizable, so this vector is minimal.

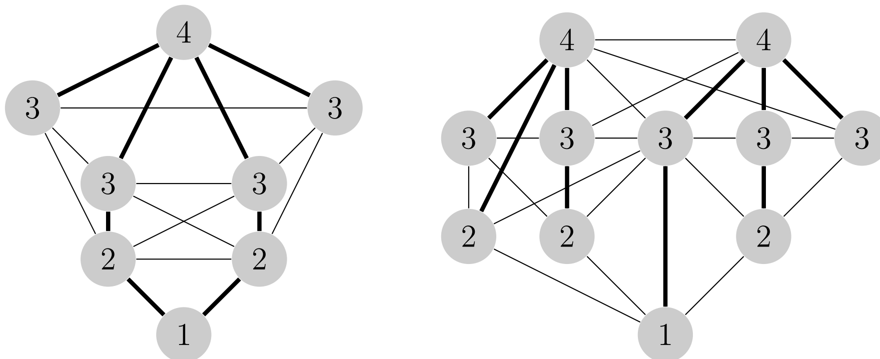

Figure 7: A graph 1-realizing (1,4,2,1) and a graph 0-realizing $(2,5,3,1)$

3. The graph $\mathbf{H}_7$ 1-realizes $(2,2,2,1)$. Theorem 3.21, Corollary 3.18, and Lemma 3.19 imply that $(1,2,2,1)$, $(2,1)$, and $(2,2,1)$ are not realizable, so $(2,2,2,1)$ is minimal.

$\square$

We now consider 0-minimal vectors; note that to be 0-realizable, the first entry must be at least 2.

Theorem 5.9. *The following vectors are* 0*-minimal.*

1. $(2,2)$
2. $(2,5,3,1)$
3. $(2,6,2,1)$
4. $(3,3,2,1)$

Proof.

1. The path $\mathbf{P}_4$ 0-realizes of $(2,2)$. Corollary 3.18 implies $(2,1)$ is not realizable, so $(2,2)$ is minimal.
2. See Figure 7 for a graph that 0-realizes $(2,5,3,1)$. The vector is minimal since $(2,4,3,1)$ and $(2,5,2,1)$ are not 0-realizable, by Theorem 3.25.
3. See Figure 8 for a graph that 0-realizes $(2,6,2,1)$. The vector is minimal since $(2,5,2,1)$ and $(2,6,1)$ are not 0-realizable by Theorem 3.25 and Lemma 3.19, respectively.
4. See Figure 8 for a graph that 0-realizes of $(3,3,2,1)$. The vector is minimal since $(2,3,2,1)$, $(3,2,2,1)$, and $(3,3,1)$ are not 0-realizable by Theorem 3.25, Theorem 3.26, and Lemma 3.19, respectively.

$\square$

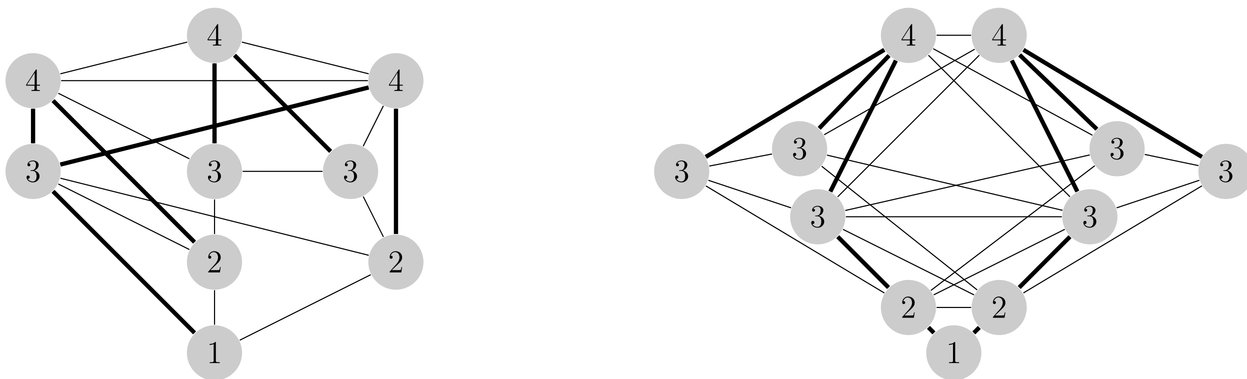

Figure 8: Graphs that 0-realize $(3,3,2,1)$ and $(2,6,2,1)$.

5.2 Capture Time $n-5$

While our understanding of minimal vectors is incomplete, we know enough to prove Theorem 5.11, our second main theorem.

Theorem 5.10. *Let $x_\alpha,\dots,x_1$ have the property that $x_j = 3$ for some $j > 1$, and $x_i = 2$ for all $i \neq j$. Then*

(i) There is exactly one graph that realizes $(1, x_\alpha, \dots, x_1)$.

(ii) $(1, x_\alpha, \dots, x_1, 1)$ is not *realizable.*

(iii) There is exactly one graph that 0-realizes $(x_\alpha, \dots, x_1)$.

(iv) $(x_\alpha, \dots, x_1, 1)$ is not *0-realizable.*

Proof.

Proof of (i):

We first suppose we have a vector $\overline{\mathbf{x}}$ of the form $(1,3,2)$ or $(1,2,\dots,2,3,2)$, and we will show that it is uniquely realized, so we let $\mathbf{G}$ be this unique graph. If $\overline{\mathbf{x}}$ is $(1,3,2)$ or $(1,2,3,2)$, we will show that the corresponding graph $\mathbf{G}$ is drawn in Figure 9. Otherwise, we are considering an $\overline{\mathbf{x}}$ of length at least 5, of the form $(1,2,\dots,2,3,2)$; in this case the corresponding graph $\mathbf{G}$ is partially drawn on the right side of Figure 9: its bottom four ranks are drawn; also there are no edges between $V(\mathbf{G}^{[5]})$ and any vertex of rank less than 4. Once we have shown that such a vector $\overline{\mathbf{x}}$ corresponds to such a unique graph $\mathbf{G}$, we can quickly obtain the uniqueness claim for any vector which is a standard extension of $\overline{\mathbf{x}}$. Considering any such standard extension of $\overline{\mathbf{x}}$, using the properties of $\mathbf{G}$, and key facts like *Path Contraction*, we can see that any such standard extension is only realized by attaching an appropriate length path to each of the rank 1 vertices of $\mathbf{G}$. The bulk of the proof now consists in showing that vectors of the form $\overline{\mathbf{x}}$ are uniquely realized in the manner described.

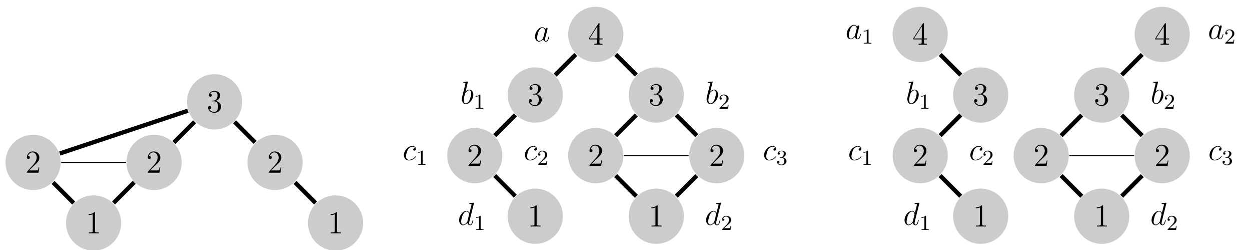

Figure 9: The unique graphs realizing $(1,3,2)$ and $(1,2,3,2)$, and the four lowest ranks of the unique graph realizing $(1,2,\ldots,2,2,3,2)$.

We first deal with the cases of $(1,3,2)$ and $(1,2,3,2)$. It is a simple exercise to see there is only one graph that realizes $(1,3,2)$ (see Figure 9). Now we show that there is only one graph that realizes $(1,2,3,2)$. Suppose $\mathbf{G}$ realizes $(1,2,3,2)$, with $\mathsf{V}_4=\{a\}$, $\mathsf{V}_3=\{b_1,b_2\}$, $\mathsf{V}_2=\{c_1,c_2,c_3\}$ and $\mathsf{V}_1=\{d_1,d_2\}$. There are 4 graphs realizing $(1,2,3)$ (note to the reader: in finding them, note that two have an edge between a and V_1, and two do not). In each of the 4 graphs we can assume without loss of generality that b_1 is adjacent only to a and c_1, and c_1 has degree 1. Thus c_1 is at distance at least 3 from any other rank 2 vertex of $\mathbf{G}$, and in any realization of $(1,2,3,2)$, c_1 must be adjacent to a vertex d_1 that is not adjacent to b_1, c_2 or c_3. This implies c_1 must strictly corner d_1. The vertex d_2 must be adjacent to c_2 and c_3, and the only way to fill in the rest of the edges leads to Figure 9 (to help see this, note that neither b_2 nor a can strictly corner d_2).

We now consider the case where $\mathbf{G}$ is a graph of rank at least 5 that realizes $(1,2,\ldots,2,3,2)$. Let $\mathsf{V}_4=\{a_1,a_2\}$, $\mathsf{V}_3=\{b_1,b_2\}$, $\mathsf{V}_2=\{c_1,c_2,c_3\}$ and $\mathsf{V}_1=\{d_1,d_2\}$; we will show, without loss of generality, that the graph induced by these vertices of rank 4 and less, is pictured in Figure 9, on the right side, and that there are no edges between $\mathbf{G}^{[5]}$ and the vertices of rank less than 4. By Theorem 3.21,

($\bigstar$) $\mathbf{G}^{[3]}$ is uniquely realized as a path.

By ($\star$), and without loss of generality, a_1 is adjacent to b_1, a_2 is adjacent to b_2, and the distance between b_1 and b_2 in $\mathbf{G}^{[3]}$ is at least 4. Thus by *Path Contraction*, the distance between b_1 and b_2 in $\mathbf{G}$ is at least 4. Thus b_1 and b_2 cannot share any neighbors of rank 2, so without loss of generality we can assume b_1 is adjacent to c_1 but not c_2 and b_2 is adjacent to c_2, but not c_1. We now make an *observation:*

If the only rank 2 neighbor of b_i is c_i, then b_i must strictly corner c_i in $\mathbf{G}^{[2]}$.

Consider why the observation is true. Since c_i is adjacent to b_i, by ($\star$), the only vertices that could strictly corner c_i in $\mathbf{G}^{[2]}$ are a_i and b_i. If a_i strictly cornered c_i in $\mathbf{G}^{[2]}$ then it would also strictly corner b_i in $\mathbf{G}^{[2]}$, which cannot happen, so b_i must strictly corner c_i in $\mathbf{G}^{[2]}$. So the observation is true.

As mentioned above, at most one of b_1 or b_2 can be adjacent to c_3, so for some i, the only rank 2 neighbor of b_i is c_i. Thus the shortest path in $\mathbf{G}^{[2]}$ between c_1 and c_2 must include

b_i and a_i, so by *Path Contraction*, c_1 and c_2 cannot be adjacent, nor adjacent to the same vertex. Thus without loss of generality, d_1 is adjacent to c_1 and not c_2, and d_2 is adjacent to c_2 and not c_1.

Now c_3 must be adjacent to one of the rank 1 vertices, without loss of generality d_2. Since c_2 and c_3 are at distance at most 2 in $\mathbf{G}$, by *Path Contraction*, in $\mathbf{G}^{[2]}$ they are at distance at most 2, from which we can conclude that there is a vertex x in $\mathbf{G}^{[3]}$ that is adjacent to both c_2 and c_3 (note that if c_2 and c_3 were adjacent, then the vertex x will be the vertex that strictly corners c_3 in $\mathbf{G}^{[2]}$). We show that b_2 must be adjacent to c_3, by assuming for contradiction that it were not. Then by the *observation*, b_2 must strictly corner c_2 in $\mathbf{G}^{[2]}$, so a_2 is not adjacent to c_2 and so cannot be x. By assumption, x is not b_2. Since b_2 strictly corners c_2 in $\mathbf{G}^{[2]}$, b_2 has to be adjacent to x violating $(\star)$. So we have that b_2 is adjacent to both c_2 and c_3. Thus, just as we argued that c_2 is not adjacent to d_1, so c_3 is not adjacent to d_1.

Now, by $(\star)$, only a_2 or b_2 can strictly corner either c_2 or c_3 in $\mathbf{G}^{[2]}$, but since a_2 cannot be adjacent to both c_2 and c_3, b_2 must strictly corner at least one of c_2 and c_3; without loss of generality, assume b_2 strictly corners c_3 in $\mathbf{G}^{[2]}$. Now consider what vertex y strictly corners d_2. The vertex y would have to be adjacent to at least d_2, c_2, and c_3. We know $y \neq a_2$ since a_2 cannot be adjacent to both c_2 and c_3. The vertex y cannot be another vertex in $\mathbf{G}^{[4]}$, since then y would be adjacent to c_3 and since b_2 strictly corners c_3 in $\mathbf{G}^{[2]}$, b_2 would have to be adjacent to y, violating $(\star)$. The vertex y can also not be b_2 since then b_2 would in fact strictly corner c_3 in $\mathbf{G}$. Thus d_2 is strictly cornered by one of c_2 or c_3, meaning that c_2 is adjacent to c_3. Viewing Figure 9, we have shown that all the displayed edges must be there and have ruled out most of the missing edges; we just need to rule out a few more edges. We rule out any other edges attached to c_2 by considering what could corner c_2 in $\mathbf{G}^{[2]}$: not a_2 since then a_2 would be adjacent to c_2 and c_3, and not any other vertex in $\mathbf{G}^{[4]}$, since by $(\star)$ it is not adjacent to b_2. So only b_2 can strictly corner c_2 in $\mathbf{G}^{[2]}$, so there can be no more edges attached to c_2. We rule out an edge between d_1 and d_2 using *Path Contraction*, since by the reasoning to this point we can now conclude that the distance between c_1 and c_2 is at least 5 in $\mathbf{G}^{[2]}$. Also d_2 can have no neighbors besides c_2 and c_3 because if it did, then nothing could strictly corner it; similarly, d_1 can have no other neighbors besides c_1.

Proof of (iii): The argument is the same as the one for (i), with 0-realizations of $(3,2)$, $(2,3,2)$ and $(2,\ldots,2,3,2)$ in place of $(1,3,2)$, $(1,2,3,2)$, and $(1,2,\ldots,2,3,2)$.

Proofs of (ii) and (iv): Assume for contradiction that we had a graph $\mathbf{G}$ realizing the appropriate vector. Thus $\mathbf{G}^{[2]}$ is as described in parts (i) and (iii), so the two rank 2 vertices of $\mathbf{G}$ are at distance greater than 2 in $\mathbf{G}^{[2]}$, but by Corollary 3.16, must both be adjacent to the rank 1 vertex in $\mathbf{G}$, contradicting *Path Contraction*. □

Theorem 5.11. *A cop-win graph on $n \geq 11$ vertices has capture time $n-5$ if and only if one of the following conditions holds:*

- *It 1-realizes a standard extension of* $(1,4,2,1)$.
- *It* 1*-realizes a vector formed by taking a standard extension of* $(2,2,2,1)$ *and then augmenting by adding 1 to any single entry.*

- *It* 0*-realizes a standard extension of* $(3, 3, 2, 1)$*.*

Proof. By Theorem 2.3 we know that any graph satisfying one of the conditions does have capture time $n-5$, so it remains to show that we have not missed any graphs. Let $\mathbf{G}$ be a cop-win graph on $n \geq 11$ vertices, with capture time $n-5$, with rank cardinality vector $\overline{\mathbf{x}} = (x_\alpha, \ldots, x_1)$. Since $n \geq 11$, $\overline{\mathbf{x}}$ must have length at least 6, and at least one of the first 6 entries of $\overline{\mathbf{x}}$, besides x_α, must be a 1 (since otherwise Theorem 2.3 would imply $\mathbf{G}$ has capture time less than $n-5$). So suppose $x_i = 1$ and $x_j > 1$ for $i < j < \alpha$, and note that $i \leq \alpha - 3$ by Corollary 3.18 and Lemma 3.19. Consider cases on whether $\mathbf{G}$ is 0-cop-win or 1-cop-win.

- *Case:* $\mathbf{G}$ *is* 0*-cop-win.*

  If x_i is $x_{\alpha-5}$, then in order to have capture time $n-5$, we must have $(2, 2, 2, 2, 2, 1)$ as an initial segment of $\overline{\mathbf{x}}$, but this vector is not 0-realizable by Theorem 3.21.

  If x_i is $x_{\alpha-4}$, then in order to have capture time $n-5$, we have the following possible initial segments of $\overline{\mathbf{x}}$: $(3, 2, 2, 2, 1)$, $(2, 3, 2, 2, 1)$, $(2, 2, 3, 2, 1)$, or $(2, 2, 2, 3, 1)$. The first three vectors are not 0-realizable by Theorem 5.10. We can show the vector $(2, 2, 2, 3, 1)$ is not 0-realizable using Theorem 3.21 and *Path Contraction*.

  If x_i is $x_{\alpha-3}$, then in order to have capture time $n-5$, the possible initial segments are: $(3, 3, 2, 1)$, $(3, 2, 3, 1)$, $(2, 3, 3, 1)$, $(4, 2, 2, 1)$, $(2, 4, 2, 1)$, or $(2, 2, 4, 1)$. The first vector $(3, 3, 2, 1)$ is 0-realizable as required; we show that the rest are not 0-realizable. The vectors $(2, 3, 3, 1)$, $(2, 4, 2, 1)$, $(2, 2, 4, 1)$ are not realizable by Theorem 3.25. The vectors $(3, 2, 3, 1)$ and $(4, 2, 2, 1)$ are not 0-realizable by Theorem 3.26.

- *Case:* $\mathbf{G}$ *is* 1*-cop-win.*

  If x_i is $x_{\alpha-5}$, then in order to have capture time $n-5$, we must have $(1, 2, 2, 2, 2, 1)$ as an initial segment of $\overline{\mathbf{x}}$, but this vector is not realizable by Theorem 3.21.

  If x_i is $x_{\alpha-4}$, then in order to have capture time $n-5$, we have the following possible initial segments of $\overline{\mathbf{x}}$: $(1, 3, 2, 2, 1)$, $(1, 2, 3, 2, 1)$, or $(1, 2, 2, 3, 1)$. By Theorem 5.10 the first two are not realizable. We can show the vector $(1, 2, 2, 3, 1)$ is not realizable using Theorem 3.21 and *Path Contraction*.

  If x_i is $x_{\alpha-3}$, then in order to have capture time $n-5$, the possible initial segments are: $(1, 4, 2, 1)$, $(1, 2, 4, 1)$, or $(1, 3, 3, 1)$. The first vector is realizable as required. The other two are not realizable by Corollary 3.24.

□

5.3 Minimal Known Realizable Vectors

Earlier in this section we motivated Question 3.3 and went on to partially answer it. While we believe a completely satisfying answer may be difficult to find, in Theorems 5.12 and 5.13

we list some vectors which we conjecture to be minimal. However as of now we cannot prove that they are minimal. For example, in Theorem 5.12 we show that the vector $(1, 2, 8, 4, 1)$ is 1-realizable, while by Lemma 5.10, the vector $(1, 2, 3, 2, 1)$ is *not* realizable. Thus there is a "gap" in our actual knowledge: for any vector $\overline{\mathbf{y}}$ such that $(1, 2, 3, 2, 1) < \overline{\mathbf{y}} < (1, 2, 8, 4, 1)$, we do not know whether or not $\overline{\mathbf{y}}$ is realizable. We have similar gaps for the other vectors of Theorems 5.12 and 5.13.

We point out an important technical issue in finding minimal vectors. First consider 0-minimal vectors. Recall that the vector $(2, 2)$ is 0-minimal and that 0-minimal vectors cannot begin with a 1. Thus the rest of the 0-minimal vectors must end with a 1. Now consider the 1-minimal vectors. Recall that $(2, 2, 2, 1)$ and $(1, 2)$ are 1-minimal. Thus the rest of the 1-minimal vectors must start with a 1 and end with a 1. In summary, to fully answer Question 3.3, the remaining work comes down to finding:

1. The 0-minimal vectors ending with a 1, and
2. The 1-minimal vectors starting with a 1 and ending with a 1.

Theorems 5.12 and 5.13 make some steps in this direction, determining that certain vectors are realizable. For the proofs, it suffices to exhibit graphs realizing the vectors. We draw two such graphs, but in the interest of saving space, have not described the other relevant graphs here. However we have a complete collection of drawings and edge lists of these graphs in a separate document we have put up on the arXiv [7] and we have a Maple document with a program that verifies the corner ranking of these graphs (see [8]). See Figure 10 for a graph realizing the vector $(1, 3, 3, 3, 3, 2, 2, 1)$ (from Theorem 5.12) and also a graph 0-realizing the vector $(2, 4, 4, 2, 1)$ (from Theorem 5.13).

Theorem 5.12. *The following vectors are* 1*-realizable:*

$(1, 2, 8, 4, 1)$	$(1, 2, 4, 4, 4, 2, 2, 1)$	$(1, 3, 5, 4, 2, 1)$
$(1, 2, 6, 4, 2, 1)$	$(1, 2, 4, 2, 4, 2, 2, 2, 1)$	$(1, 3, 4, 4, 2, 2, 1)$
$(1, 2, 5, 4, 3, 2, 1)$	$(1, 2, 3, 3, 3, 3, 2, 2, 2, 1)$	$(1, 3, 3, 3, 3, 2, 2, 1)$
$(1, 2, 5, 3, 3, 2, 2, 1)$		

Theorem 5.13. *The following vectors are 0-realizable:*

$(2, 4, 4, 2, 1)$	$(2, 3, 3, 3, 3, 2, 2, 2, 1)$
$(2, 4, 3, 4, 2, 2, 1)$	$(3, 2, 4, 2, 3, 2, 2, 1)$
$(2, 4, 2, 4, 2, 2, 2, 1)$	$(4, 2, 4, 2, 1)$

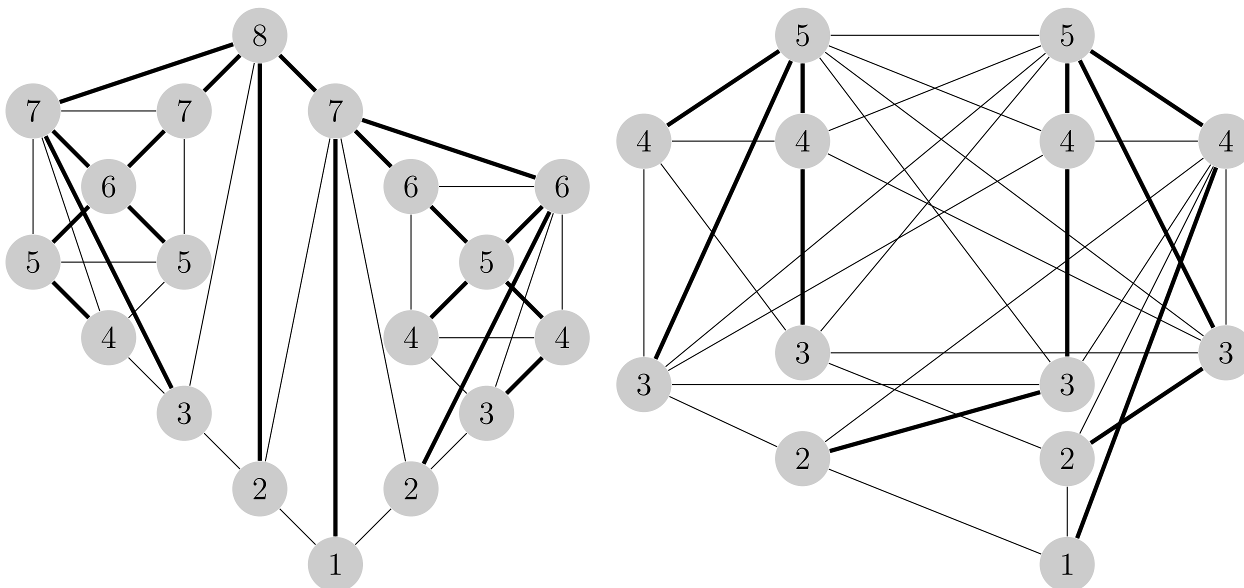

Figure 10: A graph 1-realizing $(1, 3, 3, 3, 3, 2, 2, 1)$ and a graph 0-realizing $(2, 4, 4, 2, 1)$.

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